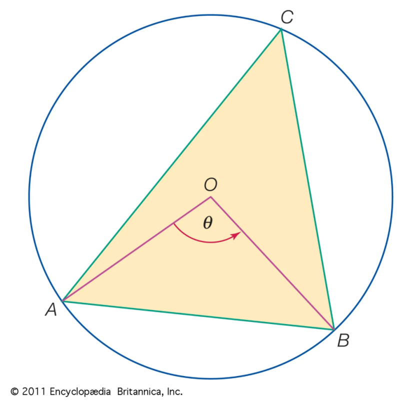
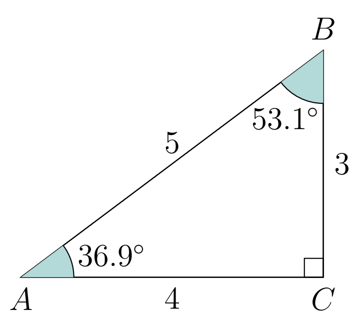
Grade 7/8 Math Circles
February 12-15, 2024
Trigonometric Ratios

What is Trigonometry?

Modern trigonometry was first introduced by the Greeks about 2,200 years ago by an astronomer named Hipparchus. He thought of every triangle as a three-sided polygon inscribed in a circle, where each side connects two points on the edge of the circle, as shown below. Since Hipparchus was an astronomer, he mainly used trigonometry to describe the position of stars and other objects in space.



Since then, trigonometry has been used for many real-world applications such as satellite navigation, chemistry, monitoring earthquakes, creating camera lenses, construction, and much more. There are countless ways to use trigonometry, even in your daily life, so having an understanding of how triangles work is an important skill. Even though we think of triangles as simple shapes, trigonometry never seems to leave us. In this lesson, we will first deal with *right-angled triangles*. A right-angled triangle is a closed polygon with three sides, and an angle of 90° somewhere inside the shape. Here is an example:



This triangle is what we call "solved". A triangle is solved when we know everything there is to know about it. First, we know all of the angles inside the triangle. We know one angle is about 36.9°, another is about 53.1°, and the last is exactly 90° denoted by the square (☐) symbol in the bottom right corner of the triangle. **All angles in a triangle must add to** 180°. Since this triangle has a 90° angle inside, it is called a *right-angled triangle*. Next, we know all of the side lengths: 3, 4, and 5. Now, what if we only knew the 90° (right) angle, and not the other two? We can still solve the triangle easily using **trigonometric ratios**.

Stop and Think

What happens when we do *not* have a right triangle? Do we give up? No! There are other mathematical tools we'll learn about to help solve triangles without a right angle. Note we can divide any triangle into two right-angled triangles; this idea will be key in our results to come.

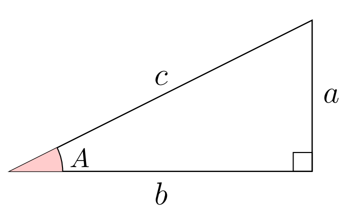
Solving Triangles

Let's have another look at the right-angled triangle above, but let's say we only know its side lengths and none of the angles other than 90°. To find more angles, we need to become familiar with the three most common trigonometric ratios: sine, cosine and tangent. We'll explore these soon.

Sides of a Right Triangle

A triangle has 3 total sides, and 2 of those sides form an angle A shown below. The longest side of a right-angled triangle is called the hypotenuse. The previous solved triangle has a hypotenuse of length 5. The other side forming an angle that is not the hypotenuse is called the adjacent side to the angle. The adjacent side of this triangle has length 4. The last side which is opposite to an angle is called the opposite side, and has length 3 in this case. Putting this all together, the side lengths of any triangle can be represented by the opposite side, adjacent side, and hypotenuse.

For the general right-angled triangle below, the angle we're looking at is A. It's opposite side length is a, the adjacent side is b, and the hypotenuse is c.



Given the lengths of any two sides of a right-angled triangle, we can find the length of the remaining side using the Pythagorean Theorem, which tells us that the sum of the squares of the opposite and adjacent sides ($a^2 + b^2$) is equal to the square of the hypotenuse (c^2). That is, for the right-angled triangle above, we relate the side lengths by the formula

$$a^2 + b^2 = c^2$$

If we just wanted to find the length of the hypotenuse c, we would take the square root of both sides.

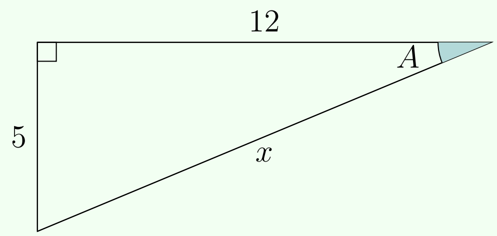
$$\sqrt{a^2 + b^2} = \sqrt{c^2}$$

$$\implies c = \sqrt{a^2 + b^2}$$

So, if we only have two sides of a right-angled triangle, don't worry! In fact, we know everything we need to know about the triangle to solve for every side length and every angle. Cool, right?

Example 1

What is the length of the unknown side x? Which sides are adjacent and opposite to angle A?



Solution:

We use the Pythagorean Theorem to find x where $a = 5$, $b = 12$ and $c = x$. So

$$(5)^2 + (12)^2 = (x)^2$$

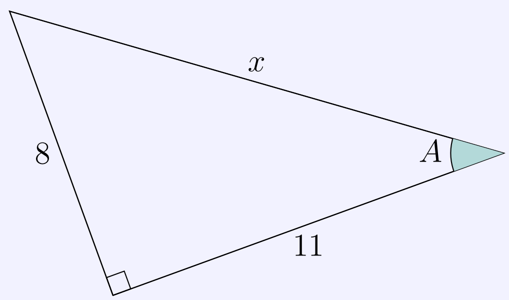
$$25 + 144 = 169 = x^2$$

$$\implies x = \sqrt{169} = 13$$

The side length opposite to angle A is 5, while the adjacent side length is 12.

Exercise 1

Consider the right-angled triangle below:



(a) Solve for the value of x.

(b) What if instead of calculating the length of the hypotenuse by the formula $a^2 + b^2 = c^2$, we changed our choice of a and b to get $b^2 + a^2 = c^2$? Does this change our answer?

Exercise 2

In the triangle from Example 1, what is the ratio of the opposite side of $\angle A$ to its hypotenuse? What about the ratio of its adjacent side to the hypotenuse? Using angles from 0° to 90°, explore the value of these ratios in this simulation and discuss why these patterns occur.

Sine, Cosine and Tangent

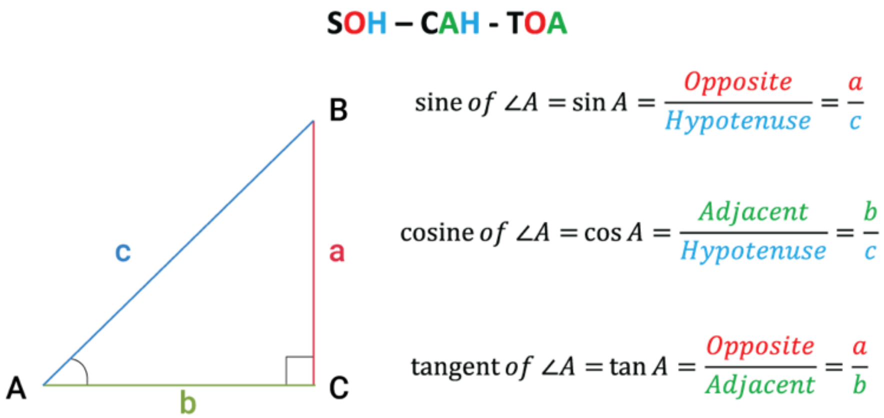
In the previous section, we learned how to find the opposite and adjacent sides from some angle A. The ratio of these side lengths are what determine the angle A; since we know all of their values, we can use the three main trigonometric ratios to find A: sine, cosine and tangent.

The sine of an angle A is written as ‘$\sin(A)$’ and is defined as the opposite side divided by the hypotenuse ($\frac{O}{H}$). Like we saw in Exercise 2, it is never greater than 1.

The cosine function is written as ‘$\cos(A)$’ and is defined as the adjacent side divided by the hypotenuse ($\frac{A}{H}$). It is never greater than 1.

Lastly, the tangent function is written as ‘$\tan(A)$’ and is defined as the opposite side divided by the adjacent side ($\frac{O}{A}$). The tangent ratio is ‘unbounded’ which means it can be any real number.

One handy way to remember these ratios is by using SOH CAH TOA:



Source

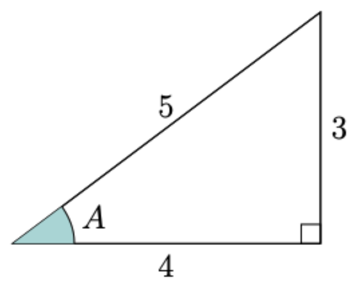
$$\sin(A) = \frac{O}{H} \qquad \cos(A) = \frac{A}{H} \qquad \tan(A) = \frac{O}{A}$$

With this in mind, let's go back to the first example we used of the right triangle with side lengths 3, 4, and 5.

The sine of angle A is the SOH in SOH CAH TOA. This means $\sin(A) = \frac{O}{H} = \frac{3}{5}$.

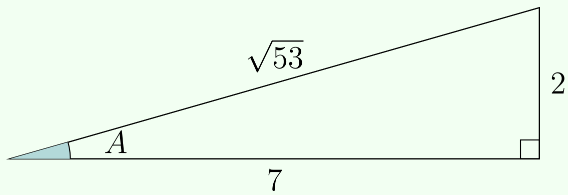
The cosine of angle A is the CAH in SOH CAH TOA. This means $\cos(A) = \frac{A}{H} = \frac{4}{5}$.

The tangent of angle A is the TOA in SOH CAH TOA. This means $\tan(A) = \frac{O}{A} = \frac{3}{4}$.



Example 2

What are the sine, cosine and tangent ratios of the angle A?



Solution:

The side length opposite (O) to angle A is 2, adjacent (A) to angle A is 7, and the hypotenuse (H) of the triangle is $\sqrt{53}$. Using SOH CAH TOA, the following ratios are

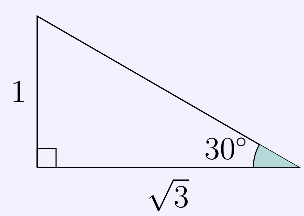
$$\sin(A) = \frac{O}{H} = \frac{2}{\sqrt{53}}$$

$$\cos(A) = \frac{A}{H} = \frac{7}{\sqrt{53}}$$

$$\tan(A) = \frac{O}{A} = \frac{2}{7}$$

Exercise 3

Given the right triangle below, what is the cosine of 30°?



Solving with Algebra & Inverse Trig Functions

Now that we know what the three basic trigonometric ratios are, how do they help us find the angle A itself? Remember that when we want to solve for variables in mathematics, we use inverse operations to eliminate the terms we don't want. For example, in the equation $3x = 9$, we want to solve for x, not $3x$. Since the 3 is multiplying the x, we recognise that the inverse operation would be to *divide* by 3. Doing this on both sides just leaves us with x on the left, which is really all we want.

$$\begin{aligned} 3x &= 9 \\ \frac{\cancel{3}x}{\cancel{3}} &= \frac{9}{3} \\ x &= 3 \end{aligned}$$

Similarly, if $\sin(A) = \frac{3}{5}$, we want to somehow get rid of the 'sin' piece to just leave us with A. To do this, we need some sort of operation that is the inverse of sine, kind of like how division is the inverse of multiplication. This function is called the 'arcsine' or 'sine-inverse', and we use it just like how we divided $3x$ by 3 — the sine and arcsine function cancel to just give us A! Note this is only true when A is *acute* (between $0°$ and $90°$), which is always true for right angled triangles.

Given that we're working with a right-angled triangle, if we know $\sin(A) = \frac{3}{5}$, we can apply the arcsin function to both sides to just get A on the left side:

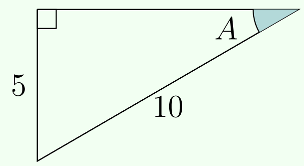
$$\begin{aligned} \sin(A) &= \frac{3}{5} \\ \cancel{\arcsin}\left(\cancel{\sin}(A)\right) &= \arcsin\left(\frac{3}{5}\right) \\ A &= \arcsin\left(\frac{3}{5}\right) \end{aligned}$$

Perfect! Now the left side is just A, and the right side is some number we can put into our calculator. The arcsine function is found by clicking the '2nd' or 'shift' button on your calculator, then the trigonometric ratio we are using, then entering the ratio value as the input. In this case, we would click '2nd $\rightarrow$ sin() $\rightarrow$ $\frac{3}{5}$' then hit Enter. Doing this gives us a value of about $37°$, which is what we saw before! The inverse of cosine is the arccosine function and acts in the same way on cosine. The same is true for tangent and arctangent. That is, when working with acute angles A:

$$\arcsin(\sin(A)) = \arccos(\cos(A)) = \arctan(\tan(A)) = A$$

Example 3

Determine the value of the acute angle A for the right-angled triangle below:



Solution:

We see that the problem involves the angle A, along with the hypotenuse of length 10 and the hypotenuse of length 5. Since we are dealing with the opposite side ($O = 5$) and the hypotenuse ($H = 10$), we will use the sine function, where $\sin(A) = \frac{O}{H}$. We solve for A by first stating

$$\sin(A) = \frac{O}{H} = \frac{5}{10} = \frac{1}{2}$$

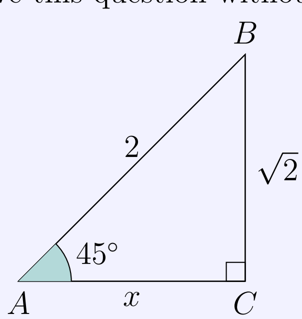
To isolate A on the left side of the equation, we take the arcsine, or inverse-sine, of both sides to simply leave us with:

$$\cancel{\arcsin}(\cancel{\sin}(A)) = \arcsin\left(\frac{1}{2}\right)$$

$$A = \arcsin\left(\frac{1}{2}\right) = 30^\circ$$

Exercise 4

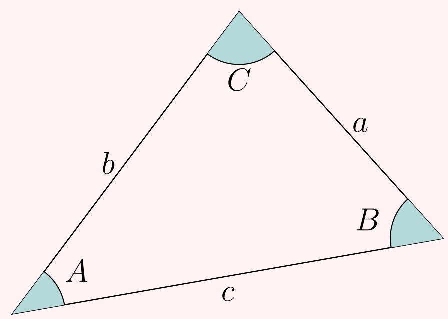
Find the side length x. Can you solve this question without a calculator? Explain.



The Sine and Cosine Law

Until now, we could only solve trigonometry problems involving right-angled triangles. Now, it's time to use what we know to help us solve any triangle we want in more realistic problems. In general, we can describe any triangle by its three angles A, B and C, and by the opposite side lengths to each angle: a, b and c, respectively. One tool we can use to solve the triangle is the Sine Law. The Sine Law tells us that the ratio of the sine of an angle in a triangle and the side length opposite to that angle is equivalent for every side-angle pair in the triangle.

Sine Law



For any triangle containing angles A, B and C with corresponding opposite side lengths a, b, and c, we can relate these values by the law:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \qquad \text{or} \qquad \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad \text{or} \qquad \frac{\sin(A)}{a} = \frac{\sin(C)}{c}$$

It's important to notice that when solving for an unknown value, we can flip both sides of the equation to modify the position of our values. This is called taking the *reciprocal* of each side of the equation.

Why can we do this?

Let's say we have the equation $\frac{2}{3} = \frac{2}{3}$. The "=" sign means that the left side of the equation is **the exact same** as the right side. If we take the reciprocal of (or 'flip') both sides, we get the obvious statement $\frac{3}{2} = \frac{3}{2}$. But, for example, when the Sine Law states that

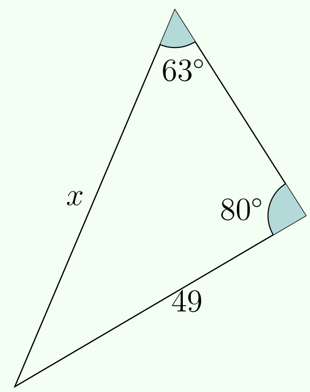
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

the "=" sign tells us that the left and right side must **always** have the same value! So, flipping both sides to get $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ is still a true statement. We can then rewrite the Sine Law as

$$\boxed{\frac{a}{\sin(A)} = \frac{b}{\sin(B)} \qquad \text{or} \qquad \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad \text{or} \qquad \frac{a}{\sin(A)} = \frac{c}{\sin(C)}}$$

Example 4

Find the value of the missing side length x of the following triangle:



Solution:

Choose $A = 63°$, $a = 49$, $B = 80°$ and $b = x$. The Sine Law tells us that

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

$$\therefore \frac{49}{\sin(63°)} = \frac{x}{\sin(80°)}$$

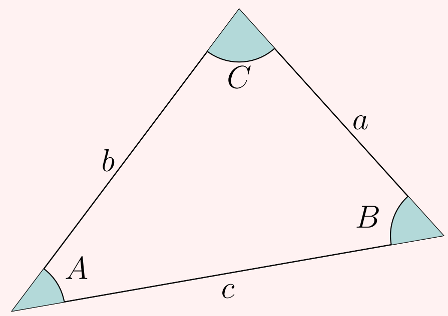
$$\sin(80°) \cdot \frac{49}{\sin(63°)} = \frac{x}{\cancel{\sin(80°)}} \cdot \cancel{\sin(80°)}$$

$$x = \sin(80°) \cdot \frac{49}{\sin(63°)} \simeq 54.2$$

Based on our diagram, this looks about right!

Keep in mind, the Sine Law is most effective when we are working with side-angle pairs like a and A, or b and B. What if we are only given side lengths a, b and c with no angles? There is one more law to help us solve this particular case the easiest: the Cosine Law.

Cosine Law



For any triangle containing angles A, B and C with corresponding opposite side lengths a, b, and c, we can relate these values by the law:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$$

where C is the angle in which side c is opposite to. We are free to choose our angles and side lengths as we please, so we can rewrite this law as

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(A) \qquad \text{or} \qquad b^2 = a^2 + c^2 - 2ac \cdot \cos(B)$$

Example 5

Imagine a triangle with side lengths $a = 9$, $b = 3$ and $c = 7$. What's the angle opposite to c?

Solution:

We want angle C. Using the Cosine Law $c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$, we plug in our values:

$$(7)^2 = (9)^2 + (3)^2 - 2(9)(3) \cdot \cos(C)$$

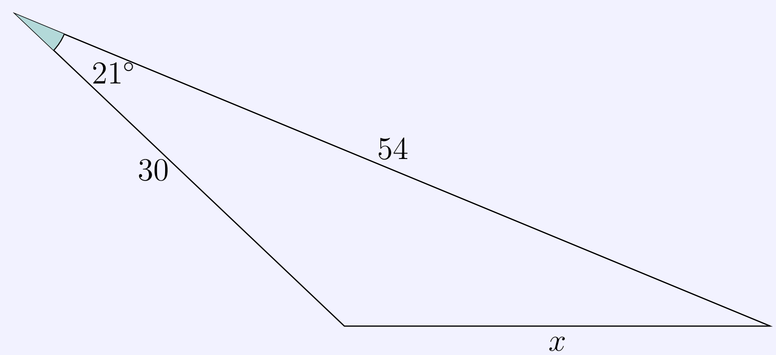
$$(7)^2 - (9)^2 - (3)^2 = -2(9)(3) \cdot \cos(C)$$

$$\cos(C) = \frac{(7)^2 - (9)^2 - (3)^2}{-2(9)(3)} \simeq 0.76 \implies C = \arccos(0.76) \simeq 40.6^\circ$$

We can also use the Cosine Law to find missing side lengths, not just angles.

Exercise 5

Find the missing side length x.



Stop and Think

For the exercise above, why did we use the Cosine Law and not the Sine Law?

Exercise 6

Measure and cut three strings of length 20cm, 15cm and 30cm to construct the triangle shown below. Ensure that the angle made between the the 20cm and 30cm string is 30°. Pivot the 15cm string about point C until its end lies along the same line as the 30cm string. Use both the Sine and Cosine Law to determine the lengths c_1 and c_2. Compare the two solutions for c.